Eur. Phys. J. B **31**, 553–561 (2003)
DOI: 10.1140/epjb/e2003-00065-y

THE EUROPEAN PHYSICAL JOURNAL B

Scaling behavior of a nonlinear oscillator with additive noise, white and colored

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Received 10 October 2002
Published online 6 March 2003 –

Abstract. We study analytically and numerically the problem of a nonlinear mechanical oscillator with additive noise in the absence of damping. We show that the amplitude, the velocity and the energy of the oscillator grow algebraically with time. For Gaussian white noise, an analytical expression for the probability distribution function of the energy is obtained in the long-time limit. In the case of colored, Ornstein-Uhlenbeck noise, a self-consistent calculation leads to (different) anomalous diffusion exponents. Dimensional analysis yields the qualitative behavior of the prefactors (generalized diffusion constants) as a function of the correlation time.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 05.10.Gg Stochastic analysis methods (Fokker-Planck, Langevin, etc.) – 05.45.-a Nonlinear dynamics and nonlinear dynamical systems

1 Introduction

The concept of noise in Physics stems from the theory of Brownian motion in which the force exerted on a macroscopic impurity by the surrounding molecules is represented by a random function $\mathcal{F}(t)$ [1]. This model provides a microscopic explanation of the law of diffusion and yields the celebrated fluctuation-dissipation relation [2]. Assuming that the Brownian particle of position $x(t)$ and velocity $v(t)$ is also subject to a harmonic potential $\mathcal{U}(x) = \frac{1}{2}\,\omega^2\,x^2$, the dynamic equation associated with this system is the linear Langevin equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) + \gamma\,v(t) + \omega^2\,x(t) = \mathcal{F}(t), \qquad (1)$$

where γ is the effective damping coefficient.

The interplay between noise and nonlinearity generates many original phenomena that make the study of random dynamical systems a rich and fascinating field. Noise in a nonlinear system may induce non-equilibrium phase transitions [3,4] or improve the performance of a device via stochastic resonance [5]. Random fluctuations may be rectified into a directed motion when a particle in a fluctuating environment is subject to a "ratchet-like" potential [6].

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A simple nonlinear noisy dynamical system is obtained by considering equation (1) with an anharmonic potential $\mathcal{U}(x)$. If the random force is approximated by a Gaussian white noise, the stationary probability distribution function (PDF) of the energy, valid when $t \gg \gamma^{-1}$, is given by the canonical distribution. To the best of our knowledge, the closed form of the distribution function of the nonlinear oscillator's energy is not known when the correlation time τ of the random force is non-zero (see [7] and references therein for approximate results valid in the small τ limit).

In this work, we study the motion of an *undamped* nonlinear oscillator subject to an additive noise. For Gaussian white noise (Sect. 2), we apply the method presented in a recent article [8], where we studied the dynamical behavior of an undamped nonlinear oscillator with a fluctuating frequency represented as a parametric noise. We calculate the probability distribution function of the energy in the long-time limit and match it to the distribution obtained in the presence of damping. The average energy, root-mean-square amplitude and velocity of the oscillator grow algebraically with time, with anomalous diffusion exponents different from those obtained for parametric noise. We show in Section 3 that diffusion is reduced at large times when the additive noise has a non-zero correlation time. Anomalous diffusion exponents are calculated in a self-consistent manner for an Ornstein-Uhlenbeck random force. In a somewhat counter-intuitive fashion, the white-noise behavior is observed for sufficiently *short* times. The crossover from white to colored noise can be embodied in a single scaling function as shown by dimensional analysis. All the results obtained for a nonlinear oscillator with additive or multiplicative, white or colored Gaussian noise are summarized in Table 1.

2 The nonlinear oscillator with additive white noise

We study a particle trapped in a confining potential $\mathcal{U}(x)$ and subject to additive noise. We neglect dissipative effects unless stated otherwise. The dynamics of the nonlinear oscillator is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) = -\frac{\partial \mathcal{U}(x)}{\partial x} + \xi(t), \tag{2}$$

where $x(t)$ represents the position of the oscillator at time t. In this section, the random noise $\xi(t)$ is a Gaussian white noise of zero mean-value and of amplitude $\mathcal{D}$:

$$\langle \xi(t) \rangle = 0\,, \qquad \langle \xi(t)\xi(t') \rangle = \mathcal{D}\,\delta(t-t'). \tag{3}$$

For analytical as well as numerical calculations, we shall interpret stochastic differential equations such as (2) according to Stratonovich's convention. Indeed, Stratonovich calculus appears naturally in the limit of a vanishingly small correlation time [2]. It must be emphasized that this convention (as opposed to Ito calculus) introduces correlations between the noise and the dynamical variables at the same time t [9]. Moreover, the mechanical oscillator defined by equations (2, 3) is *not* conservative: noise feeds energy into the system (see Eqs. (15) and (35)). In the absence of damping, the oscillator's amplitude increases with time.

For the potential $\mathcal{U}(x)$ to be confining, we must have $\mathcal{U} \to +\infty$ when $|x| \to \infty$. We shall restrict our analysis to the case where $\mathcal{U}(x)$ is a polynomial, and in order to respect the $x \to -x$ symmetry, $\mathcal{U}(x)$ is taken to be even in x. Hence, when $|x| \to \infty$,

$$\mathcal{U} \sim \frac{x^{2n}}{2n} \quad \text{with} \quad n \geq 2, \tag{4}$$

where the coefficient of x^{2n} has been set to $1/(2n)$ after a suitable rescaling. As the amplitude x of the oscillator grows at large times, only the asymptotic behavior of $\mathcal{U}(x)$ for $|x| \to \infty$ is relevant and thus equation (2) reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + x(t)^{2n-1} = \xi(t). \tag{5}$$

The method we shall use to study equations (3, 5) is akin to that presented in our previous work on nonlinear oscillators with multiplicative noise [8]. In Section 2.1, we use the integrability of the deterministic nonlinear oscillator to write exact stochastic differential equations in energy-angle variables. We then derive equipartition relations in Section 2.2. Averaging out the fast angular variable [10,11], we calculate the energy's probability distribution function in the long-time limit (Sect. 2.3). In Section 2.4, we obtain the associated anomalous diffusion exponents and constants and study their dependence on the stiffness of the confining potential at infinity. Our numerical simulations are based on a time discretization presented in [12], and described in detail in [8].

2.1 Energy-angle coordinates

We rewrite the second-order stochastic differential equation (5) as a first order system in energy (or action) and angle variables. Without noise, the deterministic version of equation (5):

$$\ddot{x} + x^{2n-1} = 0, \tag{6}$$

is integrable because of energy conservation, where the energy is defined by

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2n}x^{2n}. \tag{7}$$

The action-angle variables (I, ϕ) associated with equation (6) are

$$\begin{aligned} I &= 4\int_0^{(2nE)^{\frac{1}{2n}}} \sqrt{2E - \frac{x^{2n}}{n}}\mathrm{d}x \\ &= 4\,(2^{n+1}n)^{\frac{1}{2n}}\,E^{\frac{n+1}{2n}} \int_0^1 \sqrt{1-u^{2n}}\mathrm{d}u \\ &\propto E^{\frac{n+1}{2n}}\,, \end{aligned} \tag{8}$$

$$\phi = \sqrt{n}\int_0^{x/(2nE)^{1/2n}} \frac{\mathrm{d}u}{\sqrt{1-u^{2n}}}, \tag{9}$$

where the angle variable ϕ is defined modulo the oscillation period $4K_n$, with

$$K_n = \sqrt{n}\int_0^1 \frac{\mathrm{d}u}{\sqrt{1-u^{2n}}}\,\cdot \tag{10}$$

The solution of equation (6) can be parametrized as a function of energy E and angle ϕ

$$x = E^{1/2n}\,\mathcal{S}_n(\phi), \tag{11}$$

$$\dot{x} = (2n)^{\frac{n-1}{2n}}E^{1/2}\,\mathcal{S}_n'(\phi), \tag{12}$$

where the hyper-elliptic function $\mathcal{S}_n$ is defined as

$$\begin{aligned} \mathcal{S}_n(\phi) = Y \;\leftrightarrow \phi &= \sqrt{n}\int_0^{\frac{Y}{(2n)^{1/2n}}} \frac{\mathrm{d}u}{\sqrt{1-u^{2n}}} \\ &= \frac{\sqrt{n}}{(2n)^{1/2n}}\int_0^Y \frac{\mathrm{d}u}{\sqrt{1-\frac{u^{2n}}{2n}}}\,\cdot \end{aligned} \tag{13}$$

From this definition, we obtain the following relation between $\mathcal{S}_n$ and its derivative $\mathcal{S}'_n$

$$\mathcal{S}'_n(\phi) = \frac{(2n)^{\frac{1}{2n}}}{\sqrt{n}} \left(1 - \frac{(\mathcal{S}_n(\phi))^{2n}}{2n}\right)^{\frac{1}{2}} \cdot \tag{14}$$

Writing equation (5) in terms of energy and angle variables, we obtain the following stochastic dynamical system

$$\dot{E} = \dot{x}\,\xi(t) = (2n)^{\frac{n-1}{2n}}\, E^{\frac{1}{2}}\, \mathcal{S}'_n(\phi)\,\xi(t), \tag{15}$$

$$\dot{\phi} = (2nE)^{\frac{n-1}{2n}} - \frac{1}{(2n)^{\frac{1}{2n}}} \frac{\mathcal{S}_n(\phi)}{(2nE)^{\frac{1}{2}}}\,\xi(t). \tag{16}$$

With the help of the auxiliary variable Ω

$$\Omega = (2n)^{\frac{n+1}{2n}}\, E^{\frac{1}{2}}. \tag{17}$$

Equations (15) and (16) are written in the simpler form:

$$\dot{\Omega} = n\,\mathcal{S}'_n(\phi)\,\xi(t), \tag{18}$$

$$\dot{\phi} = \left(\frac{\Omega}{(2n)^{\frac{1}{2n}}}\right)^{\frac{n-1}{n}} - \frac{\mathcal{S}_n(\phi)}{\Omega}\,\xi(t). \tag{19}$$

These equations have been derived without any hypothesis on the function ξ and are rigorously equivalent to the original equation (5).

Although the method is identical, this set of equations differs from that obtained for parametric noise (see Eqs. (27–28) in [8]). As a consequence, additive and multiplicative random noises lead to different long-time behaviors of the observables of the system (see Tab. 1).

2.2 Equipartition relations

Noting that $\mathcal{S}_n$ and $\mathcal{S}'_n$ are bounded functions of ϕ, we deduce from equations (18) and (19) that Ω is a diffusive variable that scales as $t^{1/2}$, whereas ϕ is a fast variable that scales like $t^{3/2-1/2n}$ (these assertions will be justified rigorously in the next section). In the long time limit, it is therefore justified to average the dynamics over the variations of the fast variable ϕ. Using equations (11–12), and assuming that ϕ is uniformly distributed over the interval $[0, 4K_n]$ of a period, we find the following relations:

$$\langle E \rangle = \frac{n+1}{2n}\,\langle \dot{x}^2 \rangle, \tag{20}$$

$$\langle \dot{x}^2 \rangle = \langle x^{2n} \rangle \cdot \tag{21}$$

Figure 1 shows that equations (20–21) are indeed verified with excellent accuracy.

Because the transformation of variables (11–12) is independent of the nature of the noise $\xi(t)$, the statistical identities (20–21) are the same as those obtained for parametric noise [8]. Such generalized equipartition relations are valid whenever the asymptotic probability distribution function is independent of the angle variable and is a function of the energy only. This is the case, in particular, of the canonical Boltzmann-Gibbs distribution: equations (20–21) are valid at thermodynamic equilibrium, as can be shown using standard techniques of statistical mechanics in the canonical ensemble.

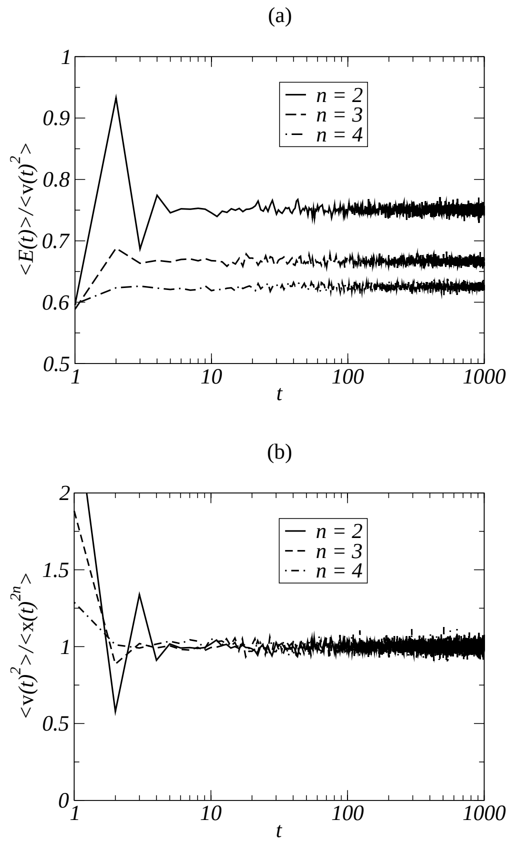

Fig. 1. Nonlinear oscillator with additive white noise: equation (5) is integrated numerically for $\mathcal{D} = 1$, with a timestep δt, and averaged over 10^4 realizations for $n = 2$ $(2n-1=3)$, $\delta t = 5 \times 10^{-4}$; $n = 3$ $(2n-1=5)$, $\delta t = 5 \times 10^{-4}$; $n = 4$ $(2n-1=7)$, $\delta t = 2 \times 10^{-4}$. (a) The first equipartition ratio $\langle E(t)\rangle/\langle v(t)^2\rangle$ is close to the theoretical value $\frac{n+1}{2n}$ given in equation (20): 3/4 for $n = 2$; 2/3 for $n = 3$; 5/8 for $n = 4$. (b) The second equipartition ratio $\langle v(t)^2\rangle/\langle x(t)^{2n}\rangle$ is close to 1 for $n = 2, 3, 4$.

2.3 Calculation of the effective PDF

The Fokker-Planck equation corresponding to the system (18, 19) reads

$$\begin{aligned}\partial_t\, P = &-\left(\frac{\Omega}{(2n)^{\frac{1}{2n}}}\right)^{(n-1)/n} \partial_\phi P \\ &+ \frac{\mathcal{D}}{2}\Bigg(\partial_\phi\Big(\frac{g(\phi)}{\Omega}\partial_\phi \frac{g(\phi)}{\Omega}P\Big) - \partial_\phi\Big(\frac{g(\phi)}{\Omega}\partial_\Omega f(\phi)P\Big) \\ &- \partial_\Omega\Big(f(\phi)\partial_\phi\frac{g(\phi)}{\Omega}P\Big) + \partial_\Omega\Big(f(\phi)\partial_\Omega f(\phi)P\Big)\Bigg)\end{aligned} \tag{22}$$

where we have defined

$$f(\phi) = n\,\mathcal{S}'_n(\phi) \quad \text{and} \quad g(\phi) = \mathcal{S}_n(\phi). \tag{23}$$

This equation governs the dynamics of the probability distribution function $P_t(\Omega, \phi)$. As we already noticed, the angle variable ϕ varies rapidly as Ω grows. We carry out the averaging of equation (22) under the hypothesis that the probability density, $P_t(\Omega,\phi)$, becomes uniform in ϕ when $t \to \infty$. The averaged evolution equation of the reduced probability density $\tilde{P}_t(\Omega)$ is then

$$\partial_t \tilde{P} = \overline{f^2(\phi)}\, \frac{\mathcal{D}}{2} \left(\partial^2_\Omega \tilde{P} - \frac{1}{n} \partial_\Omega \frac{\tilde{P}}{\Omega} \right), \tag{24}$$

where we have used $f = (\partial_\phi g)/n$. The notation $\overline{f^2(\phi)}$ represents the mean value of the function f^2 over one period $4K_n$ of ϕ. This constant has been calculated explicitly in [8] and is given by

$$\overline{f^2(\phi)} = n^2\, \overline{\mathcal{S}'_n(\phi)^2} = \frac{n^2\,(2n)^{\frac{1}{n}}}{n+1} \cdot \tag{25}$$

The Fokker-Planck equation (24) is equivalent to an effective first-order stochastic differential equation for the slow variable Ω. Defining an effective Gaussian white noise $\tilde{\xi}(t)$ such that

$$\langle \tilde{\xi}(t)\tilde{\xi}(t')\rangle = \tilde{\mathcal{D}}\,\delta(t-t') \quad \text{with} \quad \tilde{\mathcal{D}} = \frac{n^2 (2n)^{\frac{1}{n}}}{n+1}\mathcal{D}, \tag{26}$$

we deduce from equation (24) the effective Langevin equation for Ω

$$\dot{\Omega} = \frac{\tilde{\mathcal{D}}}{2\,n}\, \frac{1}{\Omega} + \tilde{\xi}(t). \tag{27}$$

Thus Ω can be reinterpreted as a Brownian variable in a logarithmic potential.

The scale-invariant effective Fokker-Planck equation (24) can be solved explicitly. We find

$$\tilde{P}_t(\Omega) = \frac{2}{\Gamma\left(\frac{n+1}{2n}\right)}\, \frac{\Omega^{\frac{1}{n}}\, \mathrm{e}^{-\Omega^2/(2\tilde{\mathcal{D}}t)}}{(2\tilde{\mathcal{D}}t)^{\frac{n+1}{2n}}}, \tag{28}$$

where Γ is the Euler Gamma function. Using equation (17), we deduce from equation (28) the asymptotic probability distribution function of the energy variable:

$$\tilde{P}_t(E) = \frac{1}{\Gamma\left(\frac{n+1}{2n}\right)}\, \frac{1}{E} \left(\frac{(2n)^{\frac{n+1}{n}} E}{2\tilde{\mathcal{D}}t} \right)^{\frac{n+1}{2n}} \times \exp\left\{ -\frac{(2n)^{\frac{n+1}{n}} E}{2\tilde{\mathcal{D}}t} \right\} \cdot \tag{29}$$

Moreover, the PDF (29) can be matched to the canonical distribution. Suppose that some damping is present in the system. The dynamical equation (5) now reads

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + \gamma \frac{\mathrm{d}}{\mathrm{d}t}x(t) + x(t)^{2n-1} = \xi(t), \tag{30}$$

where γ is the friction coefficient. The stationary distribution associated with the stochastic equation (30) is:

$$\tilde{P}_{\mathrm{can}}(E) = \frac{1}{\Gamma\left(\frac{n+1}{2n}\right)}\, \frac{1}{E} \left(\frac{2\gamma E}{\mathcal{D}} \right)^{\frac{n+1}{2n}} \exp\left\{ -\frac{2\gamma E}{\mathcal{D}} \right\} \cdot \tag{31}$$

This PDF becomes the Boltzmann-Gibbs canonical measure when supplemented with the fluctuation-dissipation relation [2]. The crossover time t_c from the asymptotic distribution function for the undamped oscillator to the Gibbs measure is found by matching equation (29) with equation (31):

$$t_c = \frac{n+1}{2n\gamma} \cdot \tag{32}$$

Thus, there are three distinct time scales involved: a fast time scale over which the angle variable ϕ gets uniformly distributed, a 'long' time scale over which the effective Fokker-Planck equation for Ω (24) is relevant, and finally times longer than the crossover time t_c beyond which the inevitable damping in the system takes over.

2.4 Time-asymptotic behavior of observables

With the probability distribution function (29), we can calculate the long time behavior, in the absence of damping, of the expectation value of any observable that depends on the position and velocity of the particle. In particular, we obtain

$$\langle x^2 \rangle = \frac{\Gamma\left(\frac{3}{2n}\right)}{\Gamma\left(\frac{1}{2n}\right)} \left(\frac{2\,n^2}{n+1}\,\mathcal{D}t \right)^{\frac{1}{n}}, \tag{33}$$

$$\langle \dot{x}^2 \rangle = \frac{n\mathcal{D}}{n+1}t, \tag{34}$$

$$\langle E \rangle = \frac{\mathcal{D}}{2}t. \tag{35}$$

From equation (33), we notice that the particle diffuses more slowly in a stiffer potential well, as one would intuitively expect. The equipartition relation between energy and velocity obtained in Section 2.2 is confirmed by equations (34, 35). For *all* potentials growing algebraically at infinity, we find that the mean value of the energy grows linearly with time, with a diffusion constant equal to $\mathcal{D}/2$ (Eq. (35)). Such a remarkable and simple behavior was not obvious *a priori*.

These predictions are confirmed by numerical simulations. The universal behavior of $\langle E \rangle$ is represented in Figure 2a. Using a numerical value of the noise amplitude $\mathcal{D} = 1$ in equation (33), we obtain

$$\text{For } n = 2,\ \ \langle x^2 \rangle = 0.552\, t^{1/2}, \tag{36}$$

$$\text{For } n = 3,\ \ \langle x^2 \rangle = 0.526\, t^{1/3}, \tag{37}$$

$$\text{For } n = 4,\ \ \langle x^2 \rangle = 0.500\, t^{1/4}. \tag{38}$$

These formulae are in excellent agreement with the numerical results displayed in Figures 2b and c.

The validity of equations (33, 34, 35) can be checked in the linear case, $n = 1$. The exact solution of the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + \omega^2 x(t) = \xi(t), \tag{39}$$

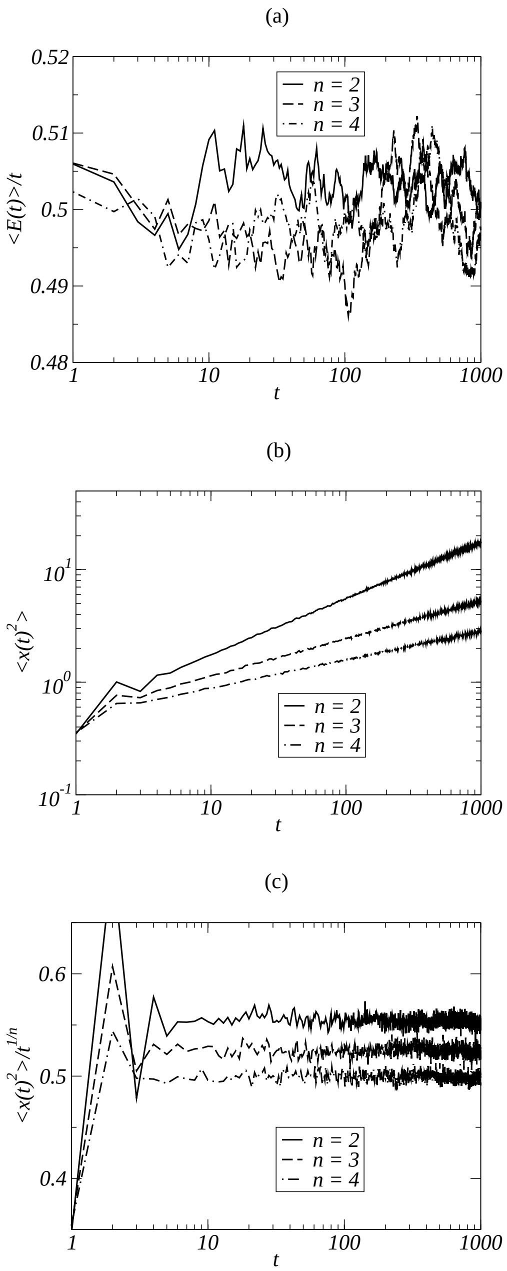

Fig. 2. Nonlinear oscillator with additive, Gaussian white noise. (a) The ratio $\langle E\rangle/t$ is found close to $\mathcal{D}/2$ ($\mathcal{D} = 1$) for $n = 2, 3, 4$. (b) Scaling behavior of $\langle x(t)^2\rangle$. (c) The ratio $\langle x^2\rangle/t^{1/n}$ yields the following generalized diffusion constants $D_x^{(n)}$: $D_x^{(2)} = 0.553(5)$, $D_x^{(3)} = 0.525(5)$, $D_x^{(4)} = 0.499(4)$, in excellent agreement with the predictions of equations (36–38). Numerical data is obtained from the same simulations as in Figure 1.

is given by

$$x(t) = \frac{1}{\omega}\int_0^t \sin\left(\omega(t-u)\right)\ \xi(u)\,\mathrm{d}u, \tag{40}$$

$$\dot{x}(t) = \int_0^t \cos\left(\omega(t-u)\right)\ \xi(u)\,\mathrm{d}u. \tag{41}$$

We obtain the following exact formulae [2]:

$$\langle x^2\rangle = \frac{\mathcal{D}}{2\omega^2}t\ \left(1 - \frac{\sin 2\omega t}{2\omega t}\right), \tag{42}$$

$$\langle \dot{x}^2\rangle = \frac{\mathcal{D}}{2}t\ \left(1 + \frac{\sin 2\omega t}{2\omega t}\right), \tag{43}$$

$$\langle E\rangle = \frac{\mathcal{D}}{2}t. \tag{44}$$

In the long time limit $t \to \infty$, these expressions agree with equations (33, 35) when one substitutes $n = 1$ (and $\omega = 1$).

3 The nonlinear oscillator with additive colored noise

In order to model an additive Gaussian noise with a non vanishing correlation time τ, we use an Ornstein-Uhlenbeck process, denoted by $\eta(t)$. We thus replace equation (5) by the following equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) + x(t)^{2n-1} = \eta(t) \tag{45}$$

where the random noise $\eta(t)$ is obtained from

$$\frac{\mathrm{d}\eta(t)}{\mathrm{d}t} = -\frac{1}{\tau}\eta(t) + \frac{1}{\tau}\xi(t), \tag{46}$$

and $\xi(t)$ is the Gaussian white noise defined in equation (3). The stationary statistical properties of η are given by

$$\langle\eta(t)\rangle = 0,$$

$$\langle\eta(t)\eta(t')\rangle = \frac{\mathcal{D}}{2\,\tau}\,\mathrm{e}^{-|t-t'|/\tau}. \tag{47}$$

We rewrite equation (45) in (Ω, ϕ) coordinates and obtain as above (see Eqs. (18) and (19))

$$\dot{\Omega} = n\,\mathcal{S}_n'(\phi)\,\eta(t), \tag{48}$$

$$\dot{\phi} = \left(\frac{\Omega}{(2n)^{\frac{1}{2n}}}\right)^{\frac{n-1}{n}} - \frac{\mathcal{S}_n(\phi)}{\Omega}\,\eta(t). \tag{49}$$

Again, the angle ϕ is a fast variable: we expect that the equipartition relations (20) and (21) are also valid for colored additive noise. This is indeed confirmed by numerical simulations (see Fig. 3).

The system (46, 48, 49) is equivalent to a Fokker-Planck equation for the PDF $P_t(\Omega, \phi, \eta)$. However, averaging this equation over ϕ does not lead to any conclusive result. The qualitative reason for this failure is as follows: when the period of the angular variable becomes smaller than the coherence time τ of the noise, the noise itself is averaged out.

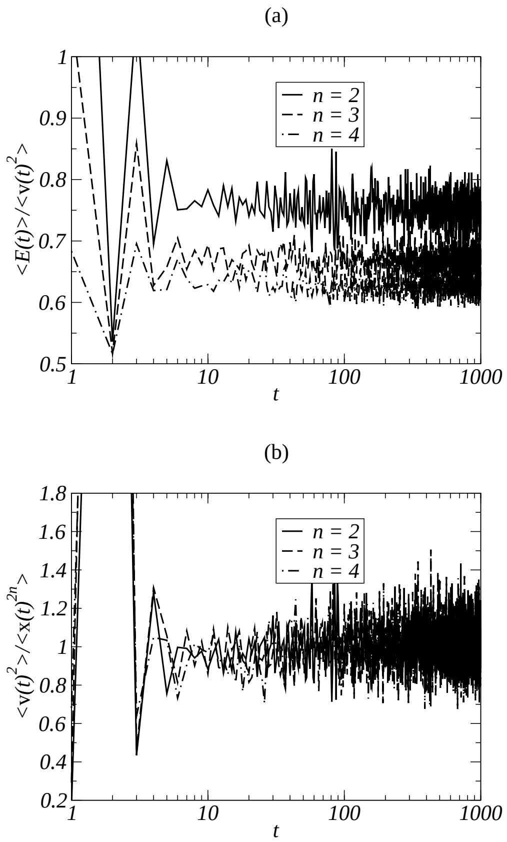

Fig. 3. Nonlinear oscillator with additive, colored noise: equations (45–46) are integrated numerically for $\mathcal{D} = 1$ and $\tau = 1$, with a timestep $\delta t = 10^{-6}$ and averaged over 500 realizations for $n = 2, 3, 4$. (a) The first equipartition ratio $\langle E(t)\rangle/\langle v(t)^2\rangle$ is close to the theoretical value $\frac{n+1}{2n}$ given in equation (20). (b) The second equipartition ratio $\langle v(t)^2\rangle/\langle x(t)^{2n}\rangle$ is close to 1.

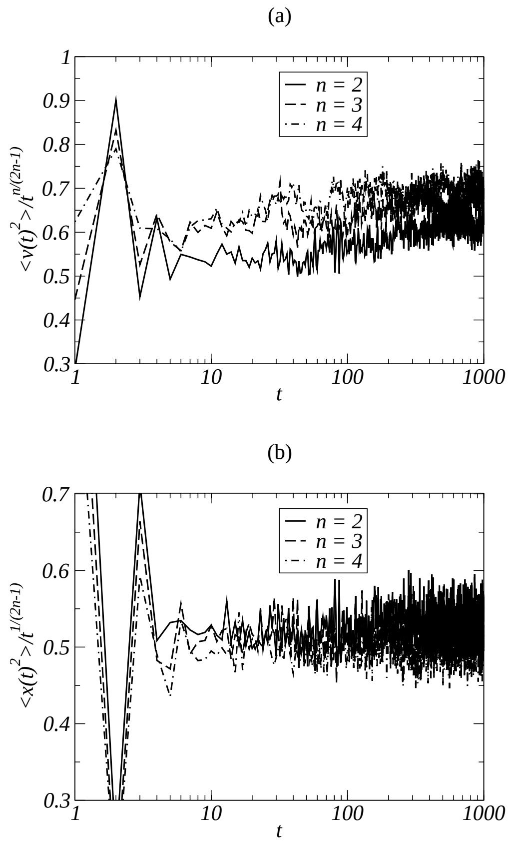

Fig. 4. Nonlinear oscillator with additive Ornstein-Uhlenbeck noise. The ratios $\langle v^2\rangle/t^{\frac{n}{2n-1}}$ (a) and $\langle x^2\rangle/t^{\frac{1}{2n-1}}$ (b) are approximately constant at large time for $n = 2, 3, 4$. Numerical data is obtained from the same simulations as in Figure 3.

The goal of this section is to introduce alternative methods to determine the scaling behavior of nonlinear oscillators with colored additive noise, first qualitatively in Section 3.1 thanks to dimensional analysis, then in a more rigorous manner in Section 3.2 using a self-consistent argument. Finally, we show in Section 3.3 that the crossover between the behaviors typical of white noise and colored noise can be embodied in a single scaling function.

3.1 Scaling analysis

The equations (45) and (46) can be interpreted as a system of three coupled first-order stochastic differential equations in presence of white noise. The corresponding Fokker-Planck equation for the PDF $P_t(x, v, \eta)$ is given by

$$\frac{\partial P}{\partial t} = -v\frac{\partial P}{\partial x} + (x^{2n-1} - \eta)\frac{\partial P}{\partial v} + \frac{1}{\tau}\frac{\partial \eta P}{\partial \eta} + \frac{\mathcal{D}}{2\tau^2}\frac{\partial^2 P}{\partial \eta^2}\,. \quad (50)$$

If we perform a scaling analysis in the spirit of [13] by comparing terms in this equation two by two, we find that a consistent balance is found if $v^2 \sim x^{2n}$, $x^{2n-1} \sim \eta$ and $\eta^2 \sim \mathcal{D}t/2\,\tau^2$. Eliminating η, we obtain the following scaling laws

$$E \sim \left(\frac{\mathcal{D}t}{2\,\tau^2}\right)^{\frac{n}{(2n-1)}},$$

$$\dot{x} \sim \left(\frac{\mathcal{D}t}{2\,\tau^2}\right)^{\frac{n}{2(2n-1)}},$$

$$x \sim \left(\frac{\mathcal{D}t}{2\,\tau^2}\right)^{\frac{1}{2(2n-1)}}. \quad (51)$$

A similar argument was used in [8] in the case of colored *multiplicative* noise, and yielded different exponents (see Tab. 1).

Numerical simulations confirm the scalings predicted by equations (51) for general n (see Fig. 4). Besides, these scalings are consistent with exact results found in the linear case $n = 1$. In the next section we present an analytic derivation of the exponents appearing in equations (51).

3.2 A self-consistent calculation

We make the *a priori* ansatz that, in the long-time limit, Ω grows algebraically with time as

$$\Omega \sim t^{\alpha}, \tag{52}$$

and we will determine the scaling exponent α from a self-consistent calculation. We deduce from equation (49) that $\phi \sim t^{\nu}$ where

$$\nu = \frac{n-1}{n}\alpha + 1. \tag{53}$$

Substituting this scaling of ϕ in equation (48), we obtain (hereafter we shall leave aside all proportionality constants)

$$\Omega \sim \int_0^t \mathrm{d}z\ \mathcal{S}_n'(z^{\nu})\,\eta(z). \tag{54}$$

We must now determine the asymptotic statistical behavior of this expression in the $t \to \infty$ limit. In order to simplify the calculation, we replace the Ornstein-Uhlenbeck noise by a discrete dichotomous noise, where time is discretized in steps of duration τ (see [14] for a general discussion). This approximation amounts to replacing exponentially decaying correlations, by finite time correlations and therefore leaves the diffusion exponents unchanged. Equation (54) then reduces to

$$\Omega \sim \sum_{k=0}^{t/\tau} \epsilon_k \int_{k\tau}^{(k+1)\tau} \mathrm{d}z\ \mathcal{S}_n'(z^{\nu}), \tag{55}$$

where the variable ϵ_k takes the value $\pm\sqrt{\mathcal{D}/(2\tau)}$ randomly during the time interval $[k\tau\,,(k+1)\tau]$; in other terms,

$$\langle \epsilon_k \epsilon_l \rangle = \frac{\mathcal{D}}{2\tau}\delta_{kl}. \tag{56}$$

From equations (55) and (56), we deduce that

$$\langle \Omega^2 \rangle \sim \sum_{k=0}^{t/\tau} \left(\int_{k\tau}^{(k+1)\tau} \mathrm{d}z\ \mathcal{S}_n'(z^{\nu}) \right)^2. \tag{57}$$

Integrating by parts we obtain

$$\int_{k\tau}^{(k+1)\tau} \mathrm{d}z\ \mathcal{S}_n'(z^{\nu}) = \frac{\mathcal{S}_n\Big((k+1)^{\nu}\tau^{\nu}\Big)}{\nu\Big((k+1)\tau\Big)^{\nu-1}} - \frac{\mathcal{S}_n\Big(k^{\nu}\tau^{\nu}\Big)}{\nu\Big(k\tau\Big)^{\nu-1}} + \frac{\nu-1}{\nu}\int_{k\tau}^{(k+1)\tau} \mathrm{d}z\ \frac{\mathcal{S}_n(z^{\nu})}{z^{\nu}}\,\cdot \tag{58}$$

The integral term on the r.h.s. of (58) is of order $\mathcal{O}(k^{-\nu})$. Since $\mathcal{S}_n$ is a bounded function, the first two terms are of order $\mathcal{O}(k^{-\nu+1})$, and will dominate when $k \gg 1$ since $\nu > 1$. We thus obtain

$$\langle \Omega^2 \rangle \sim \sum_{k=1}^{t/\tau} \frac{1}{\Big(k\tau\Big)^{2\nu-2}} \sim t^{3-2\nu}. \tag{59}$$

Assuming that the variable Ω is not multifractal, we know from the scaling hypothesis, equation (52), that $\Omega^2 \sim t^{2\alpha}$ Comparing with equation (59), we deduce that

$$2\alpha = 3 - 2\nu. \tag{60}$$

Equations (53) and (60) provide the required self-consistent condition and allow us to calculate the exponents:

$$\alpha = \frac{n}{2(2n-1)} \quad \text{and} \quad \nu = \frac{5n-3}{2(2n-1)}\,\cdot \tag{61}$$

We can now deduce from equation (17) and from the equipartition relations, equations (20) and (21), the scaling behavior of the dynamical variables

$$E \sim t^{\frac{n}{2n-1}},$$

$$v \sim t^{\frac{n}{2(2n-1)}},$$

$$x \sim t^{\frac{1}{2(2n-1)}}. \tag{62}$$

This result is in agreement with the scaling analysis of the previous section (see Eq. (51)).

3.3 Crossover from white to colored noise

We shall now match the white and the colored noise behaviors obtained in the long-time limit through a scaling function and will describe the crossover from the first regime to the second one. In the system (45) with Ornstein-Uhlenbeck noise, one can define two dimensionless variables: t/τ and $\mathcal{D}\tau^{\frac{3n-1}{n-1}}$. Any dimensionless quantity can be written as a function of these two variables (or of any two independent combinations of these two variables). For example, the mean value of the energy can be written in full generality, as a function of time, as

$$\langle E \rangle = \frac{\mathcal{D}t}{2}\,\Phi\Big(\mathcal{D}\tau^{\frac{2n}{n-1}}t,\,\mathcal{D}\tau^{\frac{3n-1}{n-1}}\Big), \tag{63}$$

where the prefactor $\mathcal{D}t/2$ ensures that the equation is dimensionally correct and has been chosen to match the white noise limit. The scaling function Φ allows us to interpolate between white and colored noise and exhibits the following asymptotic behavior:

– The white noise limit corresponds to $\tau = 0$, therefore $\Phi(0,0) = 1$.
– When τ is finite (colored noise case) and $t \to \infty$, we deduce from equation (51) that, for $u \to \infty$ and for v finite, $\Phi(u,v) \to u^{\frac{1-n}{2n-1}}\phi(v)$. The prefactor $\phi(v)$ is independent of time and cannot be determined by dimensional analysis.

The crossover between white noise and colored noise is obtained when $u = \mathcal{D}\tau^{\frac{2n}{n-1}} t \sim 1$, *i.e.*, for a typical time of the order of

$$t_{\text{cross}} \sim \frac{\tau^{-\frac{2n}{n-1}}}{\mathcal{D}} \cdot \qquad (64)$$

At this crossover time the energy is of the order of $E \sim \mathcal{D}t_{\text{cross}} \sim \tau^{-\frac{2n}{n-1}}$. The period of the deterministic oscillator for such an energy is given by $T \sim E^{-\frac{n-1}{2n}} \sim \tau$ (see Eq. (16)). Thus, the crossover occurs when the period of the deterministic oscillator is of the order of the correlation time of the noise. This fact has the following intuitive explanation. When $t \ll t_{\text{cross}}$, the period of the deterministic oscillator is very large compared to the correlation time of the noise: the noise is totally uncorrelated over one period and can be viewed as a sequence of uncorrelated kicks and thus acts as a white noise. However, when $t \gg t_{\text{cross}}$, the period of the deterministic oscillator is small compared to the correlation time of the noise: the noise is highly correlated over a period and can be viewed as an almost constant quantity over this time scale. Many periods must be added before the effect of the noise becomes perceptible: hence the diffusion slows down.

We remark that in equation (63) we have not used t/τ as the dimensionless time variable which would have been the most natural choice. In fact, the white noise limit $\tau \to 0$, and the long time limit $t \to \infty$ with finite τ are indistinguishable if one uses the variable t/τ. For this reason, it may be incorrectly stated that 'in the long time limit, colored noise appears as white'. We have seen in our problem that exactly the opposite is true: the effect of a non-zero correlation time becomes relevant at long times. The limit $t/\tau \to \infty$ is singular and its value depends on whether the second dimensionless variable $v = \mathcal{D}\tau^{\frac{3n-1}{n-1}}$ is equal to zero or is strictly positive. By choosing the variable $u = \mathcal{D}\tau^{\frac{2n}{n-1}} t$ as the dimensionless time variable in equation (63) rather than t/τ we avoid this difficulty because the white noise and the colored noise cases now appear as different limits.

4 Conclusion

A Hamiltonian nonlinear oscillator subject to random internal noise (additive noise) exhibits anomalous diffusion phenomena. We have calculated analytically the associated scaling exponents in the case of white noise and have obtained, in the long time limit, an explicit expression for the probability distribution function in phase space. The energy of such an oscillator grows linearly with time irrespective of the form of the confining potential. It is remarkable that even the rate of energy growth does not depend on the stiffness of the potential and is simply proportional to the amplitude of the noise. Our results also describe the behavior of an oscillator with small damping rate γ for times less than the dissipation time γ^{-1}. We have shown that the probability distribution function of the energy of the undamped oscillator can be matched with the canonical Boltzmann-Gibbs distribution when $t \sim \gamma^{-1}$.

Table 1. Anomalous diffusion exponents for an undamped noisy nonlinear oscillator.

Gaussian noise	white	colored
additive	$x \sim t^{\frac{1}{2n}}$	$x \sim t^{\frac{1}{2(2n-1)}}$
	$\dot{x} \sim t^{\frac{1}{2}}$	$\dot{x} \sim t^{\frac{n}{2(2n-1)}}$
	$E \sim \frac{\mathcal{D}}{2}t$	$E \sim t^{\frac{n}{(2n-1)}}$
multiplicative	$x \sim t^{\frac{1}{2(n-1)}}$	$x \sim t^{\frac{1}{4(n-1)}}$
	$\dot{x} \sim t^{\frac{n}{2(n-1)}}$	$\dot{x} \sim t^{\frac{n}{4(n-1)}}$
	$E \sim t^{\frac{n}{(n-1)}}$	$E \sim t^{\frac{n}{2(n-1)}}$

In the case of colored noise, the scaling exponents are modified. Their values have been determined by a self-consistent calculation and the predictions of dimensional analysis have been confirmed. The diffusion exponents are reduced because the coherence of the noise over a period of the system makes the energy transfer less efficient. The growth of energy is slower than linear and in the limit of a very stiff potential, the energy grows only as the square-root of time.

In Table 1, we summarize and compare the results that we have derived for additive noise and multiplicative noise, in this work and in [8], respectively. In the case of white noise, precise results are available: thanks to the averaging technique, the asymptotic probability distribution functions are known. For colored noise, only the anomalous diffusion exponents have so far been calculated. We notice that in all cases the exponent for the velocity is half the exponent for the energy and n times the exponent for the amplitude *i.e.*, the ratios between corresponding exponents are independent of the problem considered. This is the consequence of the generalized equipartition relations, which are independent of the nature of the noise (in fact, equipartition also provides universal relations between the generalized diffusion constants appearing as prefactors in the scaling laws). Besides, it was argued in [8] that, in the long time limit, the multiplicative noise always dominates over the additive noise and indeed we observe that the diffusion exponents for the multiplicative noise are always larger than those for the additive noise.

Although we have been able here to derive the exponents for additive colored noise, the generalized diffusion constants, as well as the long time asymptotics of the probability distribution function were not found by our approach and their calculation remains an open problem. Besides, the precise calculation of the scaling function that describes the crossover between white and colored noise is certainly a challenging problem.

We would like to thank Jean-Marc Luck for his constant interest in our work, as well as Michel Bauer, Peter Holdsworth and Kunihiko Kaneko for useful comments.

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